Gauge Theory of the full Lorentz Group on flat Spacetime

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Abstract

We discuss gauge theories of the Lorentz group. We discuss non-interacting, and interacting fermionic systems. The interacting system combines a local with a global Lorentz group, i.e., discusses a $SO(3,1)_l \times SO(3,1)_g$-theory. We compute the equations of motion and conservation laws for the fermionic matter current. The core of our work is the prediction of some new form of monopoles we call 'Dirac-Clifford-'t Hooft-Polyakov'-monopoles. They reside in a state similar to color-flavor locking. Dirac-Clifford-'t Hooft-Polyakov-monopoles are invariant under global Lorentz transformations and form vortices. The theory is renormalizable, since all Goldstone-Nambu modes are converted into massive vector gauge fields.

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I. INTRODUCTION

The general approach of gauging the Poincaré group has been first invented in 1961 by Kibble [1]. Our main focus is on a system containing initially two interacting fermions where globally the gauge coupling is switched off and locally been switched on, first been examined by Weinberg [2]. Our mechanism of the spontaneous breakdown of the interacting system is comparable to the color-flavor locking in quark matter [3]. In the color-flavor locking scheme the symmetry $SU(3)_{color} \times SU(3)_L \times SU(3)_R$ breaks down to the diagonal global group $SU(3)_{color+L+R}$, where color and flavor rotate simultaneously and thus are 'locked' together. The locking process results from rotating the wave-function locally into one direction and globally into the opposite direction. Color-flavor locking of quark matter comes along with the breakdown of Lorentz invariance (see [4] for a review). In our case the locking mechanism turns the system corresponding to a vacuum which is still invariant under the global Lorentz symmetry group $SO(3,1)_g$. In our spin one-half representation the Lorentz group is identified with the four dimensional rotation group $O(4)$.
Monopoles have been first predicted by Dirac [5]. They quantize the magnetic charge. 't Hooft and Polyakov invented a monopole solution based on a $SU(2)$ theory [6, 7]. In our case the matter vortices exhibit thirtytwo degrees of freedom. Based on the interaction theory $SO(3,1)_l \times SO(3,1)_g$, we predict a Dirac-Clifford-'t Hooft-Polyakov-monopole to be explained. Our theory is renormalizable, since all gauge fields become massive.
The paper is organized as follows. In sec. II we evaluate the fermionic system. In sec. III we determine the equations of motion of the fermionic system. The conservation laws are the goal of sec. IV. In sec. V. mathematical preliminaries of the interacting system are investigated. In sec. VI. we examine the physics of the interacting system. The paper finishes with conclusions in sec. VII.

II. PRELIMINARIES

A. The unitary Gauge Transformation

Let

$$\phi_\alpha \rightarrow \phi'_\alpha = \mathcal{U}_l \phi_\alpha \tag{1}$$

be a Dirac spinor where the index α indicates the particular component of the Dirac spinor. $\mathcal{U}_l$ denotes a local, unitary transformation. Let furthermore

$$\psi_\beta \rightarrow \psi'_\beta = \mathcal{U}_g \psi_\beta \tag{2}$$

be another Dirac spinor where $\mathcal{U}_g$ denotes a global, unitary transformation.

According to the transformation laws (1,2) we define a spinor product obeying the transformation law

$$\Phi_{\alpha\beta} \rightarrow \Phi'_{\alpha\beta} = \bar{\phi}_\alpha \mathcal{U}_l^\dagger \mathcal{U}_g \psi_\beta,$$

and describing a $SO(3,1)_l \times SO(3,1)_g$-system. The unitary transformation are assumed to be given by [9]

$$\mathcal{U}_l \;=\; \exp\left[\frac{i}{2}\varepsilon^{ab}(x_\mu)\mathscr{J}_{ab}\right], \tag{3a}$$

$$\mathcal{U}_g \;=\; \exp\left[\frac{i}{2}\varepsilon^{ab}\mathscr{J}_{ab}\right] \tag{3b}$$

$\mathscr{J}_{ab}$ indicates generators of rotations and boosts in relativistic quantum systems. The latin indices $a, b, \ldots$ denote some internal four dimensional space, comparable to isospin in Yang-Mills theories. In order to fulfill the role of being generators of rotations and boosts in relativistic systems $\mathscr{J}_{ab}$ is required to be traceless, i.e., $Tr(\mathscr{J}_{ab}) = 0, \forall a, b$, hermitian, i.e., $\mathscr{J}_{ab} = \mathscr{J}_{ab}^\dagger$ and antisymmetric, i.e., $\mathscr{J}_{ab} = -\mathscr{J}_{ba}$. It consists of six independent components. Check that the hermiticity of $\mathscr{J}_{ab}$ and the anti-symmetry of $\varepsilon^{ab}, \varepsilon^{ab}(x_\mu)$, $\mathscr{J}_{ab}$, yield unitarity. $\varepsilon^{ab}(x_\mu)$ indicates an infinitesimal small and local varying entity. Einstein's sum convention is assumed throughout the paper in case of greek and lower latin indices if the indices are located twice except as otherwise stated. Like the spacetime indices μ, ν the lower latin indices run from 0 to 3, where we use the signature $(+,-,-,-)$.

Since the unitary transformations (3) holds in general for infinitesimal small rotations and boosts for further use we expand $\mathcal{U}$ into a Taylor series and we keep only the linear term.

$$\mathcal{U} \;=\; 1 + \frac{i}{2}\varepsilon^{ab}(x_\mu)\mathscr{J}_{ab} + O(\varepsilon^2) \tag{4}$$

The commutation relations of the Lorentz generators are:

$$\begin{aligned} i\big[\mathscr{J}_{ab}, \mathscr{J}_{cd}\big] \;&=\; \eta_{bc}\mathscr{J}_{ad} - \eta_{ac}\mathscr{J}_{bd} \\ &-\; \eta_{da}\mathscr{J}_{cb} + \eta_{db}\mathscr{J}_{ca}, \end{aligned} \tag{5}$$

where η_{ab} represents the metric tensor of the system. It follows:

$$\begin{aligned}\frac{1}{4}\varepsilon^{ab}\varepsilon^{cd}[\mathscr{J}_{ab},\mathscr{J}_{cd}] &= \frac{1}{8}(\varepsilon^{ab}\varepsilon^{cd}-\varepsilon^{cd}\varepsilon^{ab})\big[\mathscr{J}_{ab},\mathscr{J}_{cd}\big]\\ &= 0.\end{aligned}\tag{6}$$

Therefore the expansion of transformations (3) factorizes, i.e.,

$$\begin{aligned}\mathcal{U}_l &= 1+\frac{i}{2}\varepsilon^{ab}(x_\mu)\mathscr{J}_{ab}\\ &= 1+i\varepsilon^{01}(x_\mu)\mathscr{J}_{01}i\varepsilon^{02}(x_\mu)\mathscr{J}_{02}\\ &\times\ i\varepsilon^{03}(x_\mu)\mathscr{J}_{03}i\varepsilon^{12}(x_\mu)\mathscr{J}_{12}\\ &\times\ i\varepsilon^{31}(x_\mu)\mathscr{J}_{31}i\varepsilon^{23}(x_\mu)\mathscr{J}_{23}.\end{aligned}$$

B. The Spin one-half Representation

We assume that $\mathcal{U}$ is given by (3) and the generators $\mathscr{J}_{ab}$ of the Lorentz group are given by

$$\mathscr{J}_{ab}\equiv J_{ab}=-i\frac{g}{4}\big[\gamma_a,\gamma_b\big],\tag{7}$$

where g represents the coupling constant. The Dirac γ matrices follow anti-commutation relations:

$$\{\gamma_a,\gamma_\beta\}=2\eta_{ab},\tag{8}$$

where $\gamma_0^2=1,\gamma_1^2=\gamma_2^2=\gamma_3^2=-1$. The $\mathscr{J}_{ab}$'s are traceless, hermitian and antisymmetric. In the spin one-half representation a generator J_{ab} is composed of Pauli matrices $\sigma_i, i=1,2,3$, i.e.:

$$J_{01}=-i\frac{g}{2}\begin{pmatrix}0&\sigma_1\\ \sigma_1&0\end{pmatrix}\qquad J_{02}=-i\frac{g}{2}\begin{pmatrix}0&\sigma_2\\ \sigma_2&0\end{pmatrix}\tag{9a}$$

$$J_{03}=-i\frac{g}{2}\begin{pmatrix}0&\sigma_3\\ \sigma_3&0\end{pmatrix}\qquad J_{23}=-\frac{g}{2}\begin{pmatrix}\sigma_1&0\\ 0&\sigma_1\end{pmatrix}\tag{9b}$$

$$J_{31}=-\frac{g}{2}\begin{pmatrix}\sigma_2&0\\ 0&\sigma_2\end{pmatrix}\qquad J_{12}=-\frac{g}{2}\begin{pmatrix}\sigma_3&0\\ 0&\sigma_3\end{pmatrix}\tag{9c}$$

III. FERMION LAGRANGIANS AND THEIR EQUATIONS OF MOTION

A. Global Case

Since the transformation is global, the system does not involve any gauge fields. The system is described by the Lagrangian:

$$\mathscr{L} = \int d^3x \Big\{ \bar{\psi}_\beta \gamma^\mu \partial_\mu \psi_\beta \Big\}. \tag{10}$$

Lagrangian (10) is invariant under the gauge group $SO(3,1)_g$.

B. Local Case

Since the system locally depends on spacetime we need to introduce a gauge field A_μ^{ab} transforming as follows:

$$A_\mu^{ab} \mathscr{J}_{ab} \to \mathcal{U}_l A_\mu^{ab} \mathscr{J}_{ab} \mathcal{U}_l^\dagger + \mathcal{U}_l \partial_\mu [\varepsilon^{ab}(x_\mu) \mathscr{J}_{ab}] \mathcal{U}_l^\dagger. \tag{11}$$

The Lagrangian is then given by:

$$\mathscr{L} \;=\; \int d^3x \Big\{ \bar{\phi}_\alpha \gamma^\mu D_\mu \phi_\alpha - \frac{1}{4} \mathscr{G}_{\mu\nu} \mathscr{G}^{\mu\nu} \Big\}, \tag{12}$$

where $\mathscr{G}_{\mu\nu}$ represents the field strength tensor. D_μ is the covariant derivative [8]:

$$D_\mu \phi_\alpha \equiv (\partial_\mu - i A_\mu^{ab} \mathscr{J}_{ab}) \phi_\alpha. \tag{13}$$

Since $\mathscr{J}_{ab} = -\mathscr{J}_{ba}$, the gauge fields are antisymmetric in a, b. We prove that the covariant derivative is invariant under the local unitary transformation:

$$\begin{aligned} (\partial_\mu \;-\; & i \mathscr{J}_{ab} A_\mu^{ab}) \phi_\alpha \to D'_\mu \phi'_\alpha \\ =\; & \partial_\mu (\mathcal{U}_l \phi_\alpha) - i \mathcal{U}_l \mathscr{J}_{ab} A_\mu^{ab} \mathcal{U}_l^\dagger \mathcal{U}_l \phi_\alpha \\ -\; & i \mathcal{U}_l \partial_\mu [\varepsilon^{ab}(x_\mu) \mathscr{J}_{ab}] \mathcal{U}_l^\dagger \mathcal{U}_l \phi_\alpha \\ =\; & \mathcal{U}_l \partial_\mu \phi_\alpha - i \mathcal{U}_l \mathscr{J}_{ab} A_\mu^{ab} \phi_\alpha \\ =\; & \mathcal{U}_l D_\mu \phi_\alpha \end{aligned} \tag{14}$$

This proves the local gauge invariance of the covariant derivative (13).

C. Equations of Motion

We construct the gauge invariant field strength $\mathscr{G}_{\mu\nu}$. In order to do so we compute the commutator

$$[\partial_\mu - i\mathscr{J}_{ab}A^{ab}_\mu, \partial_\nu - i\mathscr{J}_{cd}A^{cd}_\nu]\phi_\alpha = g\mathscr{G}_{\mu\nu}\phi_\alpha. \tag{15}$$

In order to be able to compute the commutator (15) we need the general commutation relations for the generators $\mathscr{J}_{ab}$. The commutator (15) provides

$$\begin{aligned} \mathscr{G}_{\mu\nu} &= i(\partial_\mu A^{ab}_\nu - \partial_\nu A^{ab}_\mu)\mathscr{J}_{ab} \\ &+ i\frac{g}{2}(A^{ab}_\mu A^{cd}_\nu - A^{cd}_\mu A^{ab}_\nu) \\ &\times (\eta_{bc}\mathscr{J}_{ad} - \eta_{ac}\mathscr{J}_{bd} - \eta_{da}\mathscr{J}_{cb} + \eta_{db}\mathscr{J}_{ca}). \end{aligned} \tag{16}$$

The computation of the equations of motion including the matter current j^μ_{ab} is straight forward. We compute the Euler-Lagrange equations of Lagrangian (12) with respect to the field A^{ab}_μ

$$\partial_\nu \frac{\partial \mathscr{L}}{\partial(\partial_\nu A^{ab}_\mu)} - \frac{\partial \mathscr{L}}{\partial A^{ab}_\mu} = 0. \tag{17}$$

This provides

$$\begin{aligned} \partial_\nu \mathscr{G}^{\mu\nu}\mathscr{J}_{ab} &+ \frac{g}{2}A^{cd}_\nu \mathscr{G}^{\mu\nu} \\ &\times (\eta_{bc}\mathscr{J}_{ad} - \eta_{ac}\mathscr{J}_{bd} - \eta_{da}\mathscr{J}_{cb} + \eta_{db}\mathscr{J}_{ca}) \\ &= j^\mu \mathscr{J}_{ab}, \end{aligned} \tag{18}$$

with the matter current

$$j^\mu = -\bar{\phi}_\alpha \gamma^\mu \phi_\alpha. \tag{19}$$

The Bianchi identity results in

$$\mathscr{D}_{\mu ef}\mathscr{G}^{\nu\rho} + \mathscr{D}_{\nu ef}\mathscr{G}^{\rho\mu} + \mathscr{D}_{\rho ef}\mathscr{G}^{\mu\nu} = 0 \tag{20}$$

with

$$\begin{aligned} \mathscr{D}_{\nu ab} = \partial_\nu J_{ab} + \frac{g}{2}A^{cd}_\nu \\ \times(\eta_{bc}\mathscr{J}_{ad} - \eta_{ac}\mathscr{J}_{bd} - \eta_{da}\mathscr{J}_{cb} + \eta_{db}\mathscr{J}_{ca}). \end{aligned} \tag{21}$$

D. Equations of motion for the Spin one-half Case

In the spin one-half representation (7) the gauge invariant field strength becomes

$$\begin{aligned} G_{\mu\nu} &= -\frac{i^p}{2}(\partial_\mu A_\nu^{ab} - \partial_\nu A_\mu^{ab})\gamma_a\gamma_b \\ &- g\frac{i^p}{4}(A_\mu^{ab}A_\nu^{cd} - A_\mu^{cd}A_\nu^{ab}) \\ &\times (\eta_{bc}\gamma_a\gamma_d - \eta_{ac}\gamma_b\gamma_d - \eta_{da}\gamma_c\gamma_b + \eta_{db}\gamma_c\gamma_a), \end{aligned} \tag{22}$$

where the exponent $p = 1$ if a or $b = 0$ and $p = 2$ if $a = b \neq 0$. This results in the Euler-Lagrange equations of motion

$$\begin{aligned} \partial_\nu G^{\mu\nu}\gamma_a\gamma_b &+ \frac{g}{2}A_\nu^{cd}G^{\mu\nu} \\ &\times (\eta_{bc}\gamma_a\gamma_d - \eta_{ac}\gamma_b\gamma_d - \eta_{da}\gamma_c\gamma_b + \eta_{db}\gamma_c\gamma_a) \\ &= j^\mu\gamma_a\gamma_b, \end{aligned} \tag{23}$$

IV. CONSERVATION LAWS

We write eq. (23), using (21), compactly in the form

$$D_{\nu ab}G^{\mu\nu} = j_{ab}^\mu. \tag{24}$$

We are able to compute the conservation law for the current j_{ab}^ν:

$$Tr\Big[D_{\nu ab}j_{ab}^\nu\Big] = 0. \tag{25}$$

It follows:

$$\begin{aligned} Tr\Big[D_{\nu ab}j_{ab}^\nu\Big] = Tr\Big[\partial_\nu\gamma_a\gamma_b - \frac{g}{2}A_\nu^{cd} \\ \times(\eta_{bc}\gamma_a\gamma_d - \eta_{ac}\gamma_b\gamma_d - \eta_{da}\gamma_c\gamma_b + \eta_{db}\gamma_c\gamma_a)\Big]j_{ab}^\nu \\ = 0 \end{aligned}$$

what results in

$$\partial_\nu j^\nu = 0. \tag{26}$$

The Noether charge Q is provided by

$$Q = -gTr[\int d^3x j^0(x_\mu)\gamma_a\gamma_b]. \tag{27}$$

Inserting the current (23) into (27) yields

$$Q = g. \tag{28}$$

V. MATHEMATICAL PRELIMINARIES OF THE INTERACTING SYSTEM

A. Construction of isotropic Dirac Spinors

Dirac spinors live on isotropic four dimensional manifolds. Mathematically they are Euclidean tensors. A product of two spinors, $\phi^\dagger_\alpha$ and ψ_β denotes our particular Euclidean tensor $\mathcal{T}_4 \in T_x\mathcal{M}$.

We use Cartan's method of a 'vector associated with a matrix X' [10], in order to decompose Dirac spinors products into their irreducible parts. Afterwards we apply the Fierz transformation [11] in order to change the position of a particular spinor within spinor products.

Based on the decomposition of spinors into its irreducible parts, a theorem by Cartan says that the product of two Dirac spinors $\tilde{\phi}^\dagger, \tilde{\phi}'$ provides:

$$\begin{aligned}
\tilde{\mathcal{T}}_4 &= \tilde{\phi}^\dagger_\mu \tilde{\phi}'_\nu \\
&= \tilde{\phi}^\dagger_\mu C X_0 \tilde{\phi}'_\nu + \tilde{\phi}^\dagger_\mu C X_1 \tilde{\phi}'_\nu + \tilde{\phi}^\dagger_\mu C X_2 \tilde{\phi}'_\nu \\
&+ \tilde{\phi}^\dagger_\mu C X_3 \tilde{\phi}'_\nu + \tilde{\phi}^\dagger_\mu C X_4 \tilde{\phi}'_\nu. \qquad (29)
\end{aligned}$$

C is some matrix depending on the space to be considered. X_0 is a scalar, X_1 is a four dimensional vector, X_2 is a four dimensional bivector, X_3 is a four dimensional trivector, and X_4 is a four dimensional fourvector [10]. In our case the spinor product $\bar{\psi}_\beta \phi_\alpha$ results in.:

$$\begin{aligned}
\mathcal{T}_4 &= \bar{\psi}_\beta \phi_\alpha \\
&= \bar{\psi}_\beta Y_0 \phi_\alpha + \bar{\psi}_\beta Y_1 \phi_\alpha + \bar{\psi}_\beta Y_2 \phi_\alpha \\
&+ \bar{\psi}_\beta Y_3 \phi_\alpha + \bar{\psi}_\beta Y_4 \phi_\alpha \\
&= \bar{\psi}_\beta \Big(u_0 + u_1 \sum_\mu \gamma^\mu + u_2 \sum_{\mu<\nu} \gamma^\mu \wedge \gamma^\nu \\
&+ u_3 \sum_{\mu<\nu<\rho} \gamma^\mu \wedge \gamma^\nu \wedge \gamma^\rho + u_4 \gamma_5 \Big) \phi_\alpha, \qquad (30)
\end{aligned}$$

where $\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3$ and the u's are complex numbers. $\wedge$ denotes the wedge product.

B. Interchanging Spinors in a Dirac Spinor Product

We need to apply the Fierz transformation [11] since it is a transformation that computes exchanges of spinors in quadrilinear products, or even higher order products of Dirac spinors

detailed computed in [12, 13]. Let

$$\bar{\psi}_\beta \Gamma^q_I \phi_\alpha \bar{\phi}_\alpha \Gamma^r_J \psi_\beta \tag{31}$$

be a bilinear product of Dirac spinors, where the Γ^r_I's denote particular gamma matrices or products of them. The indices q, r denote the particular component of $I, J = S, V, T, A, P$ where [13]

$$\begin{aligned}
\Gamma^1_S &= 1 && \text{(32a)}\\
\Gamma^1_V \dots \Gamma^4_V &= \gamma^\mu && \text{(32b)}\\
\Gamma^1_T \dots \Gamma^6_T &= -\frac{1}{2} J^{\mu\nu} = \sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right] && \text{(32c)}\\
\Gamma^1_A \dots \Gamma^4_A &= i\gamma^\mu \gamma_5 && \text{(32d)}\\
\Gamma^1_P &= \gamma_5. && \text{(32e)}
\end{aligned}$$

Note that terms (31) are products of spinor products, i.e., of terms of (29). The inverse Fierz identities are relations of the form (for each I)

$$\bar{\psi}_\beta \Gamma^q_I \psi_\beta \bar{\phi}_\alpha \Gamma_{qI} \phi_\alpha = \sum_J F^{-1}_{IJ} \bar{\psi}_\beta \Gamma^r_J \phi_\alpha \bar{\phi}_\alpha \Gamma_{rJ} \psi_\beta, \tag{33}$$

where

$$F_{IJ} = \begin{pmatrix} 1 & 1 & \frac{1}{2} & -1 & 1 \\ 4 & -2 & 0 & -2 & -4 \\ 12 & 0 & -2 & 0 & 12 \\ -4 & -2 & 0 & -2 & 4 \\ 1 & -1 & \frac{1}{2} & 1 & 1 \end{pmatrix} \tag{34}$$

Note, since the l.h.s. of (33) represents a Dirac scalar, the r.h.s. needs to be a Dirac scalar too. Using the decomposition of spinor products (30) we receive:

$$\begin{aligned}
\mathcal{T}^4_4 &= \bar{\psi}_\beta \Big(u^*_0 + u^*_1 \sum_\mu \gamma_\mu + u^*_2 \sum_{\mu<\nu} \gamma_\mu \wedge \gamma_\nu \\
&+ u^*_3 \sum_{\mu<\nu<\rho} \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho + u^*_4 \gamma_5 \Big) \phi_\alpha \\
&\times \bar{\phi}_\alpha \Big(u_0 + u_1 \sum_\mu \gamma^\mu + u_2 \sum_{\mu<\nu} \gamma^\mu \wedge \gamma^\nu \\
&+ u_3 \sum_{\mu<\nu<\rho} \gamma^\mu \wedge \gamma^\nu \wedge \gamma^\rho + u_4 \gamma_5 \Big) \psi_\beta,
\end{aligned} \tag{35}$$

Thus, combining the effects of the application of the spinor product decomposition (30) and the Fierz transformation (33) provides:

$$\begin{aligned}\mathcal{T}_4^4 \;=\; & \bar{\psi}_\beta(c_0^* + c_1^* \sum_\mu \gamma_\mu + c_2^* \sum_{\mu<\nu} \gamma_\mu \wedge \gamma_\nu \\ + \; & c_3^* \sum_{\mu<\nu<\rho} \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho + c_4^* \gamma_5)\phi_\alpha \\ \times \; & \bar{\phi}_\alpha(c_0 + c_1 \sum_\mu \gamma^\mu + c_2 \sum_{\mu<\nu} \gamma^\mu \wedge \gamma^\nu \\ + \; & c_3 \sum_{\mu<\nu<\rho} \gamma^\mu \wedge \gamma^\nu \wedge \gamma^\rho + c_4 \gamma_5)\psi_\beta,\end{aligned} \tag{36}$$

where c_i denote some complex numbers and c_i^* their conjugates.

VI. THE INTERACTING SYSTEM

A. Goldstone-Nambu Modes

We recall the basic idea of global spontaneous symmetry breaking. In this case it is a generalization of the Goldstone theorem [14, 15]. Recall that in such cases a given Lagrangian is invariant under a transformation $\mathcal{L}$,i.e.,

$$\mathscr{L}(\mathcal{L}\Phi) = \mathscr{L}(\Phi), \tag{37}$$

and if we assume too, that

$$\mathcal{L}\Phi = \Phi, \tag{38}$$

such a system is called degenerated.
If we are observing globally states Φ for which

$$\mathcal{L}\Phi \neq \Phi, \tag{39}$$

but (37) still holds, then we speak of Goldstone-Nambu modes.
To compute the interacting theory we start from the untransformed Lagrangian of fermionic/anti-fermionic system invariant under $O(4)_l \times O(4)_g$ given by:

$$\begin{aligned}\mathscr{L} \;=\; & \int d^3x \Big\{ \partial_\mu \Phi^\dagger_{\beta\beta} \partial^\mu \Phi_{\alpha\alpha} \\ - \; & m^2 \Phi^\dagger_{\beta\beta}\Phi_{\alpha\alpha} - \lambda \Big(\Phi^\dagger_{\beta\beta}\Phi_{\alpha\alpha} \Big)^2 \Big\}.\end{aligned} \tag{40}$$

Applying the rules of the spinor product and Fierz transformation yields Dirac numbers for every part of the Dirac product and thus the Lagrangian:

$$\begin{aligned}\mathscr{L} &= \int d^3x\Big\{\partial_\mu\Phi^{r\dagger}_{I\beta\alpha}\partial^\mu\Phi_{rI\alpha\beta}\\ &- m^2\Phi^{r\dagger}_{I\beta\alpha}\Phi_{rI\alpha\beta}-\lambda\Big(\Phi^{r\dagger}_{I\beta\alpha}\Phi_{rI\alpha\beta}\Big)^2\Big\},\end{aligned}\tag{41}$$

where $\Phi_{rI\alpha\beta}=\bar{\phi}_\alpha\Gamma_{rI}\psi_\beta, \Phi^{r\dagger}_{I\beta\alpha}=\bar{\psi}_\beta\Gamma^r_I\phi_\alpha$ is going to be rotated into the 'diagonal direction'. This is done by means of the unitary transformation (3). The mechanism is comparable to color-flavor locking [3], and finally yields, f.e.:

$$\Phi^{r\dagger}_{I\beta\alpha}=\bar{\psi}_\beta\mathcal{U}^\dagger_g\Gamma^r_I\mathcal{U}_l\phi_\alpha\to\Phi^\dagger_{\beta\alpha}=\bar{\psi}_\beta\phi_\alpha.\tag{42}$$

We omit the index 'I' from now on. The Lagrangian (41) becomes

$$\begin{aligned}\mathscr{L} &= \int d^3x\Big\{\partial_\mu\Phi^{\dagger}_{\beta\alpha}\partial^\mu\Phi_{\alpha\beta}\\ &- m^2\Phi^{\dagger}_{\beta\alpha}\Phi_{\alpha\beta}-\lambda\Big(\Phi^{\dagger}_{\beta\alpha}\Phi_{\alpha\beta}\Big)^2\Big\}.\end{aligned}\tag{43}$$

Now, for practical purposes, we introduce an alternative way to denote the wave functions $\Phi^\dagger_{\beta\beta}, \Phi_{\alpha\alpha}$. We define

$$\Phi^\dagger_{\beta\beta} \equiv \bar{\psi}_\beta\psi_\beta\equiv\chi^\dagger_i, \qquad i=1,\ldots,4, \tag{44a}$$

$$\Phi_{\alpha\alpha} \equiv \bar{\phi}_\alpha\phi_\alpha\equiv\xi_i, \qquad i=5,\ldots,8. \tag{44b}$$

The rotation matrix of (42), i.e., an element of the group $O(4)_l\times O(4)_g$, is given by:

$$b=\begin{pmatrix}\chi_1^* & -\chi_2^* & \chi_3^* & \chi_4^* & \xi_5^* & \xi_6^* & \xi_7^* & \xi_8^*\\ -\chi_2 & \chi_1 & -\chi_4 & \chi_3 & -\xi_6 & \xi_5 & -\xi_8 & \xi_7\\ \chi_3 & \chi_4^* & \chi_1 & -\chi_2^* & \xi_7 & -\xi_8^* & \xi_5 & -\xi_6^*\\ -\chi_4 & \chi_3^* & \chi_2 & \chi_1^* & \xi_8 & \xi_7 & \xi_6 & \xi_5^*\\ \chi_1^* & \chi_2^* & \chi_3 & -\chi_4^* & \xi_5 & -\xi_6^* & \xi_7^* & -\xi_8^*\\ -\chi_2 & \chi_1 & \chi_4 & \chi_3^* & \xi_6 & \xi_5^* & \xi_8 & \xi_7\\ \chi_3^* & -\chi_4^* & \chi_1 & \chi_2^* & \xi_7 & \xi_8^* & \xi_5^* & -\xi_6^*\\ \chi_4 & \chi_3 & -\chi_2 & \chi_1^* & -\xi_8 & \xi_7^* & \xi_6 & \xi_5\end{pmatrix}$$

and the matrix of the functions of the subgroup $O(4)_{l+g}$, i.e.,

$$\Phi^\dagger_{\beta\alpha} \equiv \bar{\psi}_\beta\phi_\alpha\equiv\zeta^*_i \qquad i=1,\ldots,8, \tag{45a}$$

$$\Phi_{\alpha\beta} \equiv \bar{\phi}_\alpha\psi_\beta\equiv\zeta_i \qquad i=1.\ldots,8, \tag{45b}$$

reads:

$$
g = \begin{pmatrix}
\zeta_1^* & -\zeta_2^* & \zeta_3^* & \zeta_4^* & 0 & 0 & 0 & 0 \\
-\zeta_2 & \zeta_1 & -\zeta_4 & \zeta_3 & 0 & 0 & 0 & 0 \\
\zeta_3 & -\zeta_4^* & \zeta_1 & -\zeta_2^* & 0 & 0 & 0 & 0 \\
\zeta_4 & \zeta_3^* & -\zeta_2 & \zeta_1^* & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \zeta_5^* & \zeta_6^* & \zeta_7^* & \zeta_8^* \\
0 & 0 & 0 & 0 & -\zeta_6 & \zeta_5 & -\zeta_8 & \zeta_7 \\
0 & 0 & 0 & 0 & \zeta_7 & -\zeta_8^* & \zeta_5 & -\zeta_6^* \\
0 & 0 & 0 & 0 & \zeta_8 & \zeta_7^* & \zeta_6 & \zeta_5^*
\end{pmatrix}
$$

Effectively we have:

$$
g \equiv \omega_{ij} = \begin{pmatrix} \omega_{1i} & 0 \\ 0 & \omega_{2j} \end{pmatrix} \tag{46}
$$

where $i = 1, \ldots, 8, j = 1, \ldots, 8$, and

$$
\omega_{1i} = \begin{pmatrix}
\zeta_1^* & \zeta_2^* & \zeta_3 & -\zeta_4^* \\
-\zeta_2 & \zeta_1 & \zeta_4 & \zeta_3^* \\
\zeta_3^* & \zeta_4^* & \zeta_1 & -\zeta_2^* \\
-\zeta_4 & \zeta_3 & \zeta_2 & \zeta_1^*
\end{pmatrix}
$$

$$
\omega_{2j} = \begin{pmatrix}
\zeta_5^* & \zeta_6^* & \zeta_7 & -\zeta_8^* \\
-\zeta_6 & \zeta_5 & \zeta_8 & \zeta_7^* \\
\zeta_7^* & \zeta_8^* & \zeta_5 & -\zeta_6^* \\
-\zeta_8 & \zeta_7 & \zeta_6 & \zeta_5^*
\end{pmatrix}
$$

The fourth order homotopy group of the coset yields:

$$
\pi_4((O(4)_l \times O(4)_g)/O(4)_{l+g}) = \mathbb{Z}, \tag{47}
$$

since in a general view it is a representation of the fourth homotopy group of a Stiefel manifold $V_{8,4} = O(8)/O(4)$. In order to compute (47) we have constructed a eight dimensional representation of the subgroup $O(4)$. The space of the coset $(O(4)_l \times O(4)_g)/O(4)_{l+g}$ is homogeneous, since $O(4)_{l+g}$ is a normal subgroup and the coset is transitive.

B. Differential geometric Approaches of the Systems

We examine the differential geometric nature of the local wave function. Note that in all issues concerning topology we refer to [16]. Let

$$\pi : \mathcal{B} \rightarrow \mathcal{M} \tag{48}$$

be a canonical projection of a (fiber-) bundle $\mathcal{B}$, which automatically associates at the tangent of every point $x \in \mathcal{M}$ a linear map called 'fiber over x' :

$$F_x = \pi^{-1}(x) \in T_x\mathcal{M}.$$

The total fiber space is given by:

$$\mathcal{B} = \bigcup_{x\in\mathcal{M}} F_x. \tag{49}$$

We demand that any F_x is diffeomorphic to the typical fiber F. Note that the full Lorentz group in spin one-half representation is equivalent to the four dimensional rotation group $O(4)$. Topologically we examine the map

$$\pi : \mathcal{B} \rightarrow \mathcal{M} \tag{50}$$

being later in the Anderson-Higgs case a non-canonical (fiber-) bundle $\mathcal{B}$, and the fiber over x is then

$$F_x = \pi^{-1}(x) \in L\mathcal{M},$$

where $L\mathcal{M}$ is the space of the connection.

C. Discussion of the Potential

Departing from the potential

$$V \; = \; m^2\omega_i^2 + \lambda\omega_i^4, \qquad i = 1, \ldots, 32 \tag{51}$$

where we've changed from two indices to one index, we assume the symmetry is spontaneously broken into the direction

$$\omega_0 = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus the vacuum is no longer invariant under $O(4)_l \times O(4)_g$, but under $O(4)_{l+g}$. We determine the minimum:

$$\frac{dV}{d\omega_i}|_{\omega_0} \;=\; m^2\omega_0\omega_i' + 2\lambda\omega_0^2\omega_i' = 0, \qquad i = 1\ldots, 32$$

and in case of $m^2 < 0$, the system spontaneously breaks down. The minima lie on:

$$\omega_0 = \Big(\frac{-m^2}{2\lambda}\Big)^{1/2} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{52}$$

with

$$\omega_0 = \Big(|\omega_1|^2 + |\omega_2|^2\Big)^{1/2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{53}$$

We expand the potential up to second order with respect to ω_i:

$$V(\omega_i) \;=\; V(\omega_0) + \frac{\partial V}{\partial\omega_i}|_{\omega_0}\delta\omega_i + \frac{\partial^2 V}{\partial\omega_i^2}|_{\omega_0}\delta^2\omega_i.$$

The first derivatives of the potential vanish in the minima and the system is massive, since a term

$$-\frac{\mu}{2} = \frac{\partial^2 V}{\partial\omega_i^2}|_{\omega_0}$$

exists.

We consider global-local locked fluctuations $\delta\omega_i = \omega_0 + \omega_i'$ around the local minima. The potential takes the form:

$$\begin{aligned} V \;&=\; m^2\Big[\omega_0^2 + 2\omega_0\omega_i' + \omega_i'^2\Big] \\ &+\; \lambda\Big[\omega_0^2 + 2\omega_0\omega_i' + \omega_i'^2\Big]^2 \\ &=\; V(\omega_0) + \frac{\partial V}{\partial\omega_i}|_{\omega_0} + m^2\omega_i'^2 \\ &+\; \lambda\Big[6\omega_0^2\omega_i'^2 + 4\omega_0\omega_i'^3 + \omega_i'^4\Big] \\ &=\; V_0 + (m^2 + 6\lambda\omega_0^2)\omega_i'^2 + 4\lambda\omega_0\omega_i'^3 + \lambda\omega_i'^4 \end{aligned} \tag{54}$$

The system contains mass since it contains terms quadratic in ω_i'. The equations of motion are up to linear order:

$$\partial_\mu\partial^\mu\omega_i' + \frac{\mu}{2}\omega_i' = 0, \qquad i = 1, \ldots, 32 \tag{55}$$

with $\mu/2 = -(m^2 + 6\lambda\omega_0^2)$.

The spontaneous breakdown yields 6 global Goldstone-Nambu modes, i.e., we receive:

$$64 \rightarrow 32 + 6. \tag{56}$$

On the r.h.s. of (56) we receive thirty two matter fields degree of freedom. This is due to the global-local locking mechanism.

D. Anderson-Higgs Mechanism

We examine a system endowed with an interaction of global-local locked fermions. We assume that the locked Lorentz symmetry is endowed with a set of gauge fields A_μ^{ab} and we implement the gauge fields in a Lagrangian invariant under $O(4)_{l+g}$ symmetry. That means we depart from the Lagrangian:

$$\begin{aligned}
\mathscr{L} &= \mathscr{L}_{fg} + \mathscr{L}_{fl} + \mathscr{L}_{int} \\
&= \int d^3x \Big\{ \bar{\psi}_\beta \gamma^\mu \partial_\mu \psi_\beta + \bar{\phi}_\alpha \gamma^\mu D_\mu \phi_\alpha \\
&+ (D_\mu \Phi_{\beta\alpha})^\dagger D^\mu \Phi_{\alpha\beta} - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} \\
&- m^2 \Phi^\dagger_{\beta\alpha} \Phi_{\alpha\beta} - \lambda \big(\Phi^\dagger_{\beta\alpha} \Phi_{\alpha\beta}\big)^2 \Big\},
\end{aligned} \tag{57}$$

with $\mathscr{L}_{fg}$ denotes the global fermionic part, $\mathscr{L}_{fl}$ denotes the local fermionic part and $\mathscr{L}_{int}$ denotes the fermion/anti-fermion interaction part, where

$$\begin{aligned}
D_\mu &= \partial_\mu - i J_{ab} A_\mu^{ab} \\
G_{\mu\nu} &= -\frac{i^p}{2} (\partial_\mu A_\nu^{ab} - \partial_\nu A_\mu^{ab}) \gamma_a \gamma_\beta \\
&- g \frac{i^p}{4} (A_\mu^{ab} A_\nu^{cd} - A_\mu^{cd} A_\nu^{ab}) \\
&\times (\eta_{bc} \gamma_a \gamma_d - \eta_{ac} \gamma_\beta \gamma_d - \eta_{da} \gamma_c \gamma_\beta + \eta_{db} \gamma_c \gamma_a).
\end{aligned}$$

E. Euler-Lagrange Equations of Motion

In order to extract the massive vector gauge fields, we need to apply the Anderson-Higgs mechanism. All six Goldstone-Nambu Bosons become massive vector gauge fields. Expanding the fermion/anti-fermion Lagrangian around the minima yields up to quadratic

order in the fluctuations:

$$\begin{aligned}
\mathscr{L}_{int} &= \int d^3x \Big\{ \partial_\mu \omega_i' \partial^\mu \omega_i' - V(\omega_0) \\
&- \frac{dV}{d\omega_i}|_{\omega=\omega_0} + \mu \omega_i'^2 \\
&+ i\Big(J_{ab} A_\mu^{ab} \partial^\mu - J^{ab} A_{ab}^\mu \partial_\mu \Big) (\omega_0 + \omega')_i^2 \\
&+ J^{ab} A_{ab}^\mu J_{cd} A_\mu^{cd} (\omega_0 + \omega_i')^2 \\
&+ J^{ab} (\partial_\mu A_{ab}^\mu)(\omega_0 + \omega_i')^2 - \frac{1}{4} G_{0\mu\nu} G_0^{\mu\nu} \Big\}.
\end{aligned} \tag{58}$$

We demand $J^{ab}(\partial_\mu A^\mu_{ab}) = 0$. That yields the Euler-Lagrange equations:

$$\partial_\mu^2 \omega_i' + \frac{\mu}{2} \omega_i' = 0, \qquad i = 1, \ldots, 32 \tag{59a}$$

$$\partial_\nu G_{0ab}^{\mu\nu} + M J^{cd} A_{cd}^\mu J_{ab} = 0, \tag{59b}$$

with

$$\begin{aligned}
M &= \omega_0^2, \\
\frac{\mu}{2} &= -(m^2 + 6\lambda \omega_0^2), \\
G_{0ab}^{\mu\nu} &= -\frac{1}{2} (\partial^\mu A^\nu - \partial^\nu A^\mu) \gamma_a \gamma_b.
\end{aligned}$$

The Anderson-Higgs mechanism causes the degrees of freedom to spontaneously breaks down as follows:

$$64 + 6 \times 2 \to 32 + 6 \times 3. \tag{60}$$

Since

$$\pi_4((O(4)_l \times O(4)_g)/O(4)_{l+g}) = \mathbb{Z}, \tag{61}$$

the spectrum of the magnetic flux is discrete. The basis predicts thirty two degrees of freedom, which are, in Landau's spirit, macroscopic measurable, vortices [17] with flux n. It is differently from an ordinary Abelian theory. The internal structure of the considered system is described by the fundamental homotopy group, yielding:

$$\pi_1(O(4)_l \times O(4)_g) = \mathbb{Z}_2. \tag{62}$$

The system has multiplicative quantum numbers and is non-orientable and is therefore identified as a magnetic monopole [18].

VII. CONCLUSIONS

We derived the equations of motion for fermionic non-interacting, and interacting systems. For the non-interacting local system we find the current to be

$$D_{\nu ab}G^{\mu\nu} = j^{\mu}_{ab}. \tag{63}$$

with

$$\begin{aligned} G_{\mu\nu} &= i(\partial_\mu A^{ab}_\nu - \partial_\nu A^{ab}_\mu)J_{ab} \\ &+ i\frac{g}{2}(A^{ab}_\mu A^{cd}_\nu - A^{cd}_\mu A^{ab}_\nu) \qquad (64) \\ &\times (\eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{da}J_{cb} + \eta_{db}J_{ca}) \qquad (65) \end{aligned}$$

and

$$\begin{aligned} D_{\nu ab} = \partial_\nu J_{ab} + \frac{g}{2}A^{cd}_\nu \\ \times(\eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{da}J_{cb} + \eta_{db}J_{ca}). \end{aligned} \tag{66}$$

We have examined an interacting system comparable to a color-flavor locking scheme, i.e., a Lagrangian containing a $O(4)_{l+g}$ symmetry. The most important results yields the interacting system. We conclude:

1. the fourth order homotopy group of the Stiefel manifold, i.e.:

   $$\pi_4(V_{8,4}) = \pi_4((O(4)_g \times O(4)_l)/O(4)_{l+g}) = \mathbb{Z},$$

   yields non-trivial solutions and expresses the topology change that appears through the spontaneous symmetry breakdown of the system;

2. the internal structure of the system is described by means of the fundamental homotopy group

   $$\pi_1(O(4)_l \times O(4)_g) \; = \; \mathbb{Z}_2;$$

3. the system is globally Lorentz invariant and predicts, in Landau's spirit [17], macroscopic measurable vortices which are called Dirac-Clifford-'t Hooft-Polyakov-monopoles, carrying quantized massive magnetic flux;

4. applying the Anderson-Higgs mechanism converts a system with 64 matter states and 6×2 massless gauge fields into a system containing 32 matter fields and six massive gauge vector fields, i.e.:

$$64 + 6 \times 2 \;\rightarrow\; 32 + 6 \times 3;$$

5. the theory is renormalizable since all gauge fields are becoming massive through the particular Anderson-Higgs mechanism;

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